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A Special Class of Rings

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Abstract

In this paper, a special class of rings for which an element a in R, satisfying the additional property $a^n = a^{n-1} + a^{n-1}$, for an integer $n > 1$ is introduced and several results and examples related to this special class of ringsis presented.

Keywords: special class of rings; central elements; Nilpotent element; sub-directly irreducible; semi-simple.

1 Introduction

Special classes of rings play vital role in abstract algebra and its applications. One of such special classes of rings is Boolean ring R_B for which $a^2 = a$ for all a in R_B. The Boolean ring plays a vital role in the areas of communication, computer science and engineering. The author in [8] studied a special class of rings on the notion of SS-element of a ring for which $a^2 = a + a$ for all a in the ring. In [5] and [6] also studied on the structure of SS elements of near-ring. The authors in [7] studied central SS-elements of a ring and [3] introduced the notion of partially ordered gamma near-rings and characterize their properties. The aim of this paper is to introduce a special class of rings R for which an element a in R, satisfying the additional property $a^n = a^{n-1} + a^{n-1}$, where $n > 1$ is an integer, for an integer a in R, and explore its results and examples.

Section 2 describes some important definitions and theorems that are needed throughout this paper. The main result and conclusion are presented in Section 3 and 4 respectively.

2 Preliminaries

Throughout this paper, the symbol R is used to represent a ring. Some definitions from [4] that are essential in proving our result are presented. Readers are referred to [1] and [2] for detailed concepts of rings. This section begins by recalling the following definitions:

Definition 2.1. *A ring R is called an idempotent ring if $a^2 = a$ for all a in R.*

Definition 2.2. *An element a of a ring R is called nilpotent if there exists a positive integer n, called the index, such that $a^n = 0$.*

Definition 2.3. *Let R be a ring. If there exists a positive integer n such that $na = 0_R$ for all $a \in R$, then the smallest such positive integer n is called the characteristic of R. If no such positive integer n exists, then R is said to be characteristic zero.*

Definition 2.4. *A special class of rings is a ring R with the additional property that $a^n = a^{n-1} + a^{n-1}$ for an element a in R and an integer $n > 1$.*

Note that if R is a special class of rings with unit element 1, then 0 and 2 are called trivial elements, since $0^n = 0^{n-1} + 0^{n-1}$ and $2^n = 2^{n-1} + 2^{n-1}$ for any integer $n > 1$. Other elements of the ring R are called non-trivial elements. Throughout this paper, the structure $a^3 = a^2 + a^2$ is used for an element a in R for proving the results.

Example 2.1. *Let $H = \{e, a\}$ be a cyclic group of order 2 and let $\mathbb{Z}_2 = \{0, 1\}$ be a field. Then, $\mathbb{Z}_2(H) = \{0, a, e, a + e\}$ is a group algebra with respect to the operations '+' and '.' defined by the following Table 1 and Table 2.*

Table 1: Addition operation over $\mathbb{Z}_2(H)$.

$+$	0	a	e	$a+e$
0	0	a	e	$a+e$
a	a	0	$a+e$	e
e	e	$a+e$	0	a
$a+e$	$a+e$	e	a	0

Table 2: Multiplication operation over $\mathbb{Z}_2(H)$.

$\cdot$	0	a	e	$a+e$
0	0	0	0	0
a	0	e	e	$a+e$
e	0	a	e	$a+e$
$a+e$	0	$a+e$	$a+e$	0

The elements 0 and $a+e$ of $\mathbb{Z}_2(H)$ satisfy the additional property $a^n = a^{n-1} + a^{n-1}$ for $n = 3$. Hence, $\mathbb{Z}_2(H)$ is a special class of rings. Note that 0 is a trivial element, and $a+e$ is a non-trivial element in $\mathbb{Z}_2(H)$.

Example 2.2. *Let $\mathbb{Z}_3 = \{0, 1, 2\}$ be a field, and let $G = \{g : g^2 = 1\}$ be a group. Then, $\mathbb{Z}_3(G) = \{0, 1, 2, g, 2g, 1+g, 2+g, 1+2g, 2+2g\}$ is a group algebra with respect to the operations ‘ + ‘ and ‘.‘ defined by the Table 3 and Table 4.*

Table 3: Addition operation over $\mathbb{Z}_3(G)$.

$+$	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
0	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
1	1	2	0	$1+g$	$1+2g$	$2+g$	g	$2+2g$	$2g$
2	2	0	1	$2+g$	$2+2g$	g	$1+g$	$2g$	$1+2g$
g	g	$1+g$	$2+g$	$2g$	0	$1+2g$	$2+2g$	1	2
$2g$	$2g$	$1+2g$	$2+2g$	0	g	1	2	$1+g$	$2+g$
$1+g$	$1+g$	$2+g$	g	$1+2g$	1	$2+2g$	$2g$	2	0
$2+g$	$2+g$	g	$1+g$	$2+2g$	2	$2g$	$1+2g$	0	1
$1+2g$	$1+2g$	$2+2g$	$2g$	1	$1+g$	2	0	$2+g$	g
$2+2g$	$2+2g$	$2g$	$1+2g$	2	$2+g$	0	1	g	$1+g$

Table 4: Multiplication operation over $\mathbb{Z}_3(G)$.

$\cdot$	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
2	0	2	1	$2g$	g	$2+2g$	$1+2g$	$2+g$	$1+g$
g	0	g	$2g$	1	2	$1+g$	$1+2g$	$2+g$	$2+2g$
$2g$	0	$2g$	g	2	1	$2+2g$	$2+g$	$1+2g$	$1+g$
$1+g$	0	$1+g$	$2+2g$	$1+g$	$2+2g$	$2+2g$	0	0	$1+g$
$2+g$	0	$2+g$	$1+2g$	$1+2g$	$2+g$	0	$2+g$	$1+2g$	0
$1+2g$	0	$1+2g$	$2+g$	$2+g$	$1+2g$	0	$1+2g$	$2+g$	0
$2+2g$	0	$2+2g$	$1+g$	$2+2g$	$1+g$	$1+g$	0	0	$2+2g$

Hence, it can be seen that $\mathbb{Z}_3(G)$ is a special class of rings.

3 Main Results

In this section, the main results are provided.

Theorem 3.1. *Let R be a ring with identity 1. Then, every special class of rings has zero divisors.*

Proof. Suppose a is a non-zero element in a special class of rings R. Then, $a^3 = a^2 + a^2$ for an element a in R. This implies that $a^2(a - 2) = 0$. From this it can be directly concluded that $a - 2 \neq 0$. If $a - 2 = 0$, then $a = 2$ which contradicts to the fact that a is a non-trivial element in R. Therefore, it is compulsory have $a - 2 \neq 0$. Thus, R has zero divisors. □

Theorem 3.2. *Let R and $R^{'}$ be two rings with unity 1 and $1^{'}$ respectively. Let $f : R \rightarrow R^{'}$ be a monomorphism. Then,*

(*i*) *R is a special class of rings if and only if $R^{'}$is a special class of rings.*

(*ii*) *If R is a special class of rings of characteristic 2 with $a \neq 0$, then $R^{'}$ contains nilpotent.*

Proof. (i) Let R be a special class of rings and let $a \in R$. Since $f(a) \in R^{'}$ and $a^3 = a^2 + a^2$, $f(a^3) = f(a^2 + a^2)$ is obtained. Using a monomorphism for f would yield $(f(a))^3 = (f(a))^2 + (f(a))^2$. Therefore, $R^{'}$ is a special class of rings. On the other-hand, let $f(a) \in R^{'}$. Then, $(f(a))^3 = (f(a))^2 + (f(a))^2$. Using a monomorphism for f would yield $a^3 = a^2 + a^2$. This concludes that R is a special class of rings.

(ii) Let R be a special class of rings of characteristic 2 with $a \neq 0$. Since $R^{'}$ is a special class of rings, and $f(a) \in R^{'}$, in view of (i), $(f(a))^3 = (f(a))^2 + (f(a))^2$. Then, $f(a^3) = f(a^2 + a^2) = f(2a^2)$. This gives $f(0) = (f(a))^3$. Therefore, $(f(a))^3 = 0^{'}$. Hence, $f(a)$ is nilpotent in $R^{'}$. □

Theorem 3.3. *If e_i for $i = 1, 2, 3, ..., n$ are idempotent elements and a is an element in special class of rings R, then $u = \sum_{i=1}^{n} e_i\, a$ is a element in special class of rings R.*

Proof. Let R be special class of rings and let $u = \sum_{i=1}^{n} e_i\, a$. Then,

$$u^3 = (\sum_{i=1}^{n} e_i\, a)^3 = \sum_{i=1}^{n} \overset{3}{e}_i\, a^3 = \Big(\sum_{i=1}^{n} e_i\Big)^3 a^3 = \Big(\sum_{i=1}^{n} e_i^2 e_i\Big)\, a^3.$$

Since e_i for $i = 1, 2, ..., n$ are idempotent elements, we can further write,

$$u^3 = \Big(\sum_{i=1}^{n} e_i * e_i\Big)\, a^3 = \sum_{i=1}^{n} e_i^2 (a^2 + a^2) = \sum_{i=1}^{n} e_i^2\, a^2 + \sum_{i=1}^{n} e_i^2\, a^2 = u^2 + u^2.$$

Hence, $u = \sum_{i=1}^{n} e_i\, a$ is an element in special class of rings R. □

Definition 3.1. *An element c of a special class of rings R is said to be central if $cx = xc$ for all $x \in R$.*

Theorem 3.4. *Let $\varphi : R \to R^{'}$ be ring homomorphism. If c is a central of special class of rings R, then $\varphi(c)$ is also central in a special class of rings $R^{'}$.*

Proof. Let c be a non-zero central element in a special class of rings R. By definition 10, $cx = xc$ for every $x \in R$. Since $\varphi : R \to R^{'}$ be ring homomorphism and $c, x \in R$. Then, $\varphi(c), \varphi(x) \in R^{'}$. Now,

$$\varphi(c)\varphi(x) = \varphi(cx) = \varphi(xc) = \varphi(x)\varphi(c).$$

Hence, $\varphi(c)$ is central element in the special class of rings $R^{'}$. □

Theorem 3.5. *Let a be a non-zero element in a special class of rings R, then $a^3 \neq a^2$.*

Proof. Let a be a non-zero element in special class of rings R. Suppose $a^3 = a^2$, then $a^3 = a^2 + a^2$ for $n = 3$. Then, $a^2 = 0$ gives $a = 0$, which is a contradiction. Hence, the theorem is proved. □

Theorem 3.6. *Let R be a special class of rings with unity 1 of characteristic 2, and $a \in R$ if and only if a is nilpotent with index 3.*

Proof. Let R be a special class of rings and let $a \in R$. Then, $a^3 = a^2 + a^2$ gives $a^3 = 0$. Hence, a is nilpotent of the special class of rings R with index 3. On the other hand, converse is obvious. □

Theorem 3.7. *Let R be a special class of rings with unity 1 of characteristic 2 and let c be any non-zero central element in R. If b is a non-zero element in R, then cbc is an element in R.*

Proof. By theorem 3.6, it can be seen that $c^3 = 0$. Since $c \in R$ is a central element, then $cb = bc$ for all $b \in R$. Then,

$$(cbc)^2 + (cbc)^2 = 2(cbc)^2 = 0.$$

Also,

$$(cbc)^3 = (cbc)(cbc)(cbc) = cbc^2(bc)cbc = cbc^2(cb)cbc = cbc^3(bc)(bc) = 0.$$

From the above equations, it can be seen that cbc is an element of the special class of rings R. □

Theorem 3.8. *Let R be a special class of rings with unity 1 and let a be any non-zero element of R with characteristic $\neq 2$. $(1-a)$ is an element in the special class of rings R if and only if a has its own inverse.*

Proof. Let R be special class of rings with unity 1 and let a be any non-zero element of R with characteristic $\neq 2$. Suppose $(1-a)$ is an element of R. Then,

$$(1-a)^3 = (1-a)^2 + (1-a)^2.$$

This gives that,

$$a^2 - a^3 = 1 - a.$$

By right cancellation law, this yields $a^2 = 1$. Hence, a has its own inverse. On the other hand, converse is obvious. □

Theorem 3.9. *Let $\phi : R \to Re \times R(1-e)$ is a homomorphism from a special class of rings R to another special class of rings $Re \times R(1-e)$ defined by $\phi(a) = (ae, a(1-e))$, where e and $1-e$ are idempotent elements of Re and $R(1-e)$, respectively. If R has a central, then $Re \times R(1-e)$ has a central.*

Proof. Let R and $Re \times R(1-e)$ be two special class of rings. Let e and $1-e$ are idempotent elements of Re and $R(1-e)$, respectively. Then $e^2 = e$ and $(1-e)^2 = 1-e$. Let $a \in R$ be a non-zero element in R. Then,

$$a^3 = a^2 + a^2.$$

Now,

$$(\phi(a))^3 = \phi(a^3) = \phi(a^2 + a^2) = \phi(a^2) + \phi(a^2) = (\phi(a))^2 + (\phi(a))^2.$$

Hence, $\phi(a)$ is element in $Re \times R(1-e)$. Let $c^{'}, b^{'} \in Re \times R(1-e)$ such that $c^{'} = \phi(c) = (ce, c(1-e))$ and $b^{'} = \phi(b) = (be, b(1-e))$ for every $c, b \in R$. Let c be central in R, then there exist $b \in R$ such that $cb = bc$. Now,

$$\begin{aligned} c^{'}b^{'} =& \phi(c)\phi(b) = \phi(cb) = \phi(bc) = (bce, bc(1-e)) = (bce^2, bc(1-e)^2) = (bcee, bc(1-e)(1-e)) \\ &= (b(ce)e, b(c(1-e)(1-e)) = (b(ec)e, b((1-e)c(1-e)) = (be.ce, b(1-e).c(1-e)) \\ &= ((be, b(1-e))(ce, c(1-e)) = \phi(b)\phi(c) = b^{'}c^{'}. \end{aligned}$$

Hence, $Re \times R(1-e)$ has a central element. ☐

Theorem 3.10. *Let $H = \{f/f^2 = 1\}$ be a group and let $\mathbb{Z}_p$ be a field of characteristic p, where p is a prime. Then, the group algebra $\mathbb{Z}_p(H)$ is a special class of rings.*

Proof. Let $(1+f) \in \mathbb{Z}_p(H)$. It is observed that,

$$(1+f)^2 = (1+f)(1+f) = 1 + 2f + f^2 = 2(1+f).$$

Now,

$$(1+f)^2 + (1+f)^2 = 2(1+f) + 2(1+f) = 4(1+f),$$

and

$$(1+f)^3 = (1+f)(1+f)^2 = (1+f)2(1+f) = 4(1+f).$$

Therefore,

$$(1+f)^3 = (1+f)^2 + (1+f)^2.$$

Hence, $(1+f)$ is a non-trivial element of $\mathbb{Z}_p(H)$.

Now,

$$(1+(p-1)f)^2 = 1 + 2(p-1)f + (p-1)^2 f^2 = 1 + 2(p-1)f + 1 = 2 + 2(p-1)f,$$

and

$$(1+(p-1)f)^3 = (1+(p-1)f)(1+(p-1)f)^2 = (1+(p-1)f)(2+2(p-1)f) = 4 + 4(p-1)f.$$

Hence, it can be seen that $1 + (p-1)f$ is a non-trivial element of $\mathbb{Z}_p(H)$.

If $l + mf$, $l \neq 1, m \neq 1$ are non-trivial element of $\mathbb{Z}_p(H)$, then it can be derived that $(l+mf)^3 = (l+mf)^2 + (l+mf)^2$. Expanding this equation would yield

$$(l^2 + 3lm^2) + (m^3 + 3l^2m)f = 2(l^2 + m^2) + 2(lm + ml)f.$$

By comparing coefficients of f and constants would obtain

$$l^2 + 3lm^2 = 2(l^2 + m^{2)},$$

and

$$m^3 + 3l^2m = 2(lm + ml).$$

These equations are satisfied only when $l = m = 1$. This is a contradiction to our assumption. Hence, there does not exist any element other than $(1+f)$ and $1 + (p-1)f$ to be elements of $\mathbb{Z}_p(H)$. Hence, $(1+f)$ and $(1+(p-1)f)$ are only non-trivial elements of $\mathbb{Z}_p(H)$. Therefore, $\mathbb{Z}_p(H)$ is a special class of rings. $\square$

Definition 3.2. [1] *A ring R is called (right) primitive if and only if R has a faithful irreducible module. If R is arbitrary and I is the set of irreducible $R-$modules, then the kernel of I is called the radical of R. If I is faithful, then R is called semi-simple.*

Lemma 3.1. [*Maschke Theorem*] *Let G be a finite group of order n and K be a field with a characteristic which does not divide n. Then, the group ring $K(G)$ is semi-simple.*

Theorem 3.11. *Let $\mathbb{F}(G)$ be a group algebra, where $\mathbb{F}$ is a field and G is a cyclic group generated by an element a of order $n(n$ is even integer). Then, the characteristic of a field $\mathbb{F}$ is p or $p-2$ if and only if $1 + sa^m$, where $s = p-2$ and $m = \frac{n}{2}$, is a non-trivial element of $\mathbb{F}(G)$.*

Proof. Suppose that $\mathbb{F}$ is a field of characteristic p or $p-2$. Let $\frac{n}{2} = m$ and $s = p-1$. Then it is obvious that $(1+sa^m)$ value neither zero nor 2. Now,

$$(1+sa^m)^2 = (1+sa^m)(1+sa^{m)} = 1+(p-1)(p-1)a^{2m}+((p-1)+(p-1))a^m = 2+(2p-2)a^m = 4+2pa^m,$$

since the characteristic of the field $\mathbb{F}$ is p. And,

$$(1+sa^m)^2 + (1+sa^m)^2 = (2p^2 - 4p + 4) + (4p-4)a^m = 4 - 4a^m,$$

since the characteristic of the field $\mathbb{F}$ is p. This shows that $(1+sa^m)$ is a non-trivial element of $\mathbb{F}(G)$.

Now,

$$\begin{aligned}
(1+sa^m)^3 &= (1+sa^m)(1+sa^m)^2 \\
&= (1+sa^m)((p^2-2p+2) + (2p-2)a^m) \\
&= (1+(p-1)a^m)(p^2-2p+2+(2p-2)a^m) \\
&= p^2-2p+2+(2p-2)a^m + (p-1)(p^2-2p+2)a^m + (p-1)(2p-2)a^{2m} \\
&= p^2-2p+2+[(2p-2)+(p^3-2p^2+2p-p^2+2p-2)]a^m + (2p^2-2p-2p+2) \\
&= p^2-2p+2+[p^3-3p^2+6p-4]a^m + (2p^2-4p+2) \\
&= 3p^2-6p+4+(p^3-3p^2+6p-4)a^m.
\end{aligned}$$

If the characteristic of the field $\mathbb{F}$ is p then the above equation gives $4-4a^m$. If the characteristic of the field F is $p-2$ then the above equation gives $(4+2pa^m)$. This shows that $(1+sa^m)$ is a non-trivial element of $\mathbb{F}(G)$. On the other hand, let $(1+sa^m)$ is a non-trivial element of $\mathbb{F}(G)$. Then, it can be deduced that

$$(1+sa^m)^3 = (1+sa^m)^2 + (1+sa^m)^2.$$

By comparing coefficients of a^m and constants, $1+3s^2 = 2+2s^2$ and $s^3+3s = 4s$. From these equations, we have $s^2 = 1$, so that $(p-1)^2 = 1$. This gives $p(p-2) = 0$. This implies that either $p = 0$ or $p = 2$, since $\mathbb{F}$ has no zero divisors. Similarly, it can be observed that characteristic of the field $\mathbb{F}$ is either p or $p-2$. □

Theorem 3.12. *Let $\mathbb{F}(G)$ be a group algebra, where F is a field and G is a cyclic group generated by an element a of order $n(n$ is even integer$)$. If $1+sa^m$, where $s = p-2$ and $m = \frac{n}{2}$, is a non-trivial element of $\mathbb{F}(G)$, then the group algebra $\mathbb{F}(G)$ is semi-simple.*

Proof. Suppose that $1+sa^m$ is a non-trivial element of $\mathbb{F}(G)$. Let $\mathbb{F}$ be a field and let G be a cyclic group generated by an element a of order $n(n$ is even integer$)$. By Theorem 3.11, the characteristic of the field $\mathbb{F}$ is either p or $p-2$. Consequently, the characteristic of the field $\mathbb{F}$ is an odd number and does not divide n. By Lemma 3.1, $\mathbb{F}(G)$ is semi-simple. □

4 Conclusions

In this paper, a special class of rings is introduced. Some results and examples related to the special class of rings have been presented. Furthermore, the study of this special class of rings can be extended to other branches of algebras such as near-rings, near-fields, near-modules and near-algebras.

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